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TWO EXAMPLES OF LEFSCHETZ FIXED POINT FORMULA WITH RESPECT TO SOME BOUNDARY CONDITIONS

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Abstract. The boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$ were introduced in [5] by using the Hodge decomposition on the de Rham complex. In [6] the Atiyah-Bott-Lefschetz type fixed point formulas were proved on a compact Riemannian manifold with boundary for some special type of smooth functions by using these two boundary conditions. In this paper we slightly extend the result of [6] and give two examples showing these fixed point theorems.

1. Introduction

In this paper we are going to discuss the Atiyah-Bott-Lefschetz type fixed point formula on a compact Riemannian manifold with boundary with respective to some boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$. In [5] R.-T. Huang and the author introduced new boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$ for the odd signature operator acting on the space of smooth differential forms on a compact Riemannian manifold with boundary and in [6] they proved the Atiyah-Bott-Lefschetz type fixed point formulas for a special type of smooth functions with respect to the boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$. In this paper we slightly extend the result of [6] and give two examples showing the results obtained in [6]. Hence, this paper is a continuation of [6]. For a self-contained presentation, some material in [6] will be repeated for a background explanation.

We begin with the Atiyah-Bott-Lefschetz fixed point formula on a compact closed Riemannian manifold given in [1] and its generalization to a compact Riemann manifold with boundary given in [3]. Let (M, g^M)

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be an m-dimensional compact closed Riemannian manifold and $f : M \to M$ be a smooth map. A point $x_0 \in M$ is called a simple fixed point of f if

$$f(x_0) = x_0, \qquad \det\big(\operatorname{Id} - df(x_0)\big) \neq 0. \tag{1.1}$$

If x_0 is a simple fixed point, then the graph of f is transverse to $M \times M$ at (x_0, x_0), which shows that simple fixed points are discrete. We define the Lefschetz number $L(f)$ by

$$L(f) = \sum_{q=0}^{m} (-1)^q \operatorname{Tr}\big(f^* : H^q(M) \to H^q(M)\big), \tag{1.2}$$

where $H^q(M)$ is computed with the coefficient $\mathbb{R}$. All through this paper the cohomology groups including $H^q(M)$, $H^q(Y)$ and $H^q(M, Y)$ are computed with respect to $\mathbb{R}$ so that we ignore the torsion parts. It is shown in [1] (see also [8]) that if f has only simple fixed points, then

$$L(f) = \sum_{f(x)=x} \operatorname{sign}\det\big(\operatorname{Id} - df(x)\big). \tag{1.3}$$

Let (M, Y, g^M) be a compact Riemannian manifold with boundary Y and $f : M \to M$ be a smooth map with $f(Y) \subset Y$. We assume that all fixed points of f are simple. On the boundary Y, we need one more ingredient. Let $f(x_0) = x_0$ with $x_0 \in Y$. Considering two maps $df(x_0) : T_{x_0}M \to T_{x_0}M$ and $d(f|_Y)(x_0) : T_{x_0}Y \to T_{x_0}Y$, there is a map

$$a_{x_0} := df(x_0)(\operatorname{mod} T_{x_0}Y) : T_{x_0}M/T_{x_0}Y \ \to\ T_{x_0}M/T_{x_0}Y. \tag{1.4}$$

Since $T_{x_0}M/T_{x_0}Y$ is isomorphic to $\mathbb{R}$, if follows that a_{x_0} is a 1×1 matrix, which is a real number. Considering the map $f : M \to M$ with $f(Y) \subset Y$, a_{x_0} is a linear map from $[0, \infty)$ to $[0, \infty)$ and hence a_{x_0} is identified with a non-negative real number. Since x_0 is a simple fixed point, it follows that $a_{x_0} \neq 1$. A simple boundary fixed point x_0 is called *attracting* and *repelling* if $0 \leq a_{x_0} < 1$ and $a_{x_0} > 1$, respectively. We denote by $\mathcal{F}_0(f)$, $\mathcal{F}_Y^+(f)$ and $\mathcal{F}_Y^-(f)$ the set of all interior fixed points, attracting and repelling boundary fixed points of f, respectively. We denote $\mathcal{F}_Y(f) = \mathcal{F}_Y^+(f) \cup \mathcal{F}_Y^-(f)$. It is shown in [3] that

$$\begin{aligned}(1.5)\qquad &\sum_{q=0}^{m} (-1)^q \operatorname{Tr}\big(f^* : H^q(M) \to H^q(M)\big) \\ &\qquad = \sum_{x \in \mathcal{F}_0 \cup \mathcal{F}_Y^+(f)} \operatorname{sign}\det\det\big(\operatorname{Id} - df(x)\big), \\ &\sum_{q=0}^{m} (-1)^q \operatorname{Tr}\big(f^* : H^q(M,Y) \to H^q(M,Y)\big) \\ &\qquad = \sum_{x \in \mathcal{F}_0 \cup \mathcal{F}_Y^-(f)} \operatorname{sign}\det\det\big(\operatorname{Id} - df(x)\big).\end{aligned}$$

As a reference, the Lefschetz fixed formula on a compact manifold with conical singularities was discussed in [2], which is irrelevant to this note.

On the other hand, new boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$ were introduced in [5] and similar fixed point formulas were proved in [6] for a special type of smooth functions with respect to the boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$. In the next section, we are going to review the boundary conditions $\widetilde{\mathcal{P}}_0$, $\widetilde{\mathcal{P}}_1$ and results obtained in [6]. In Section 3, we are going to give two examples showing these results.

2. The boundary conditions $\widetilde{\mathcal{P}}_0$, $\widetilde{\mathcal{P}}_1$ and de Rham complex $(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M), d)$

The material in this section is not new and most parts are found in Section 2 of [6]. However, for a self-contained presentation, we are going to review some material of Section 2 in [6]. Furthermore, we are going to complete the proofs of some results which were skipped in Section 2 of [6], and extend the results slightly, which justifies this section.

Let (M, Y, g^M) be an m-dimensional compact oriented Riemannian manifold with boundary Y. We choose a collar neighborhood U of Y which is diffeomorphic to $[0, \epsilon) \times Y$. We denote by (u, y) the coordinate of $[0, \epsilon) \times Y$ and by du, $\frac{\partial}{\partial u}$ the one form and vector field which is normal to Y on U. We assume that the metric g^M on U is the product one so that

$$(2.1)\qquad g^M|_U = du^2 + g^Y,$$

where g^Y is a Riemannian metric on $\{0\} \times Y$. We denote by d_q^Y : $\Omega^q(Y) \to \Omega^{q+1}(Y)$ the exterior derivative acting on smooth q-forms on

Y and $\star_Y$ be the Hodge star operator on Y which is induced from the Hodge star operator $\star_M$ on M by $d\,\mathrm{vol}(M) = du \wedge d\,\mathrm{vol}(Y)$. Then, the formal adjoint $\left(d_q^Y\right)^* : \Omega^{q+1}(Y) \to \Omega^q(Y)$ is defined by $\left(d_q^Y\right)^* = (-1)^{mq+q+1} \star_Y d_q^Y \star_Y$ and the Laplacian $\Delta_Y^q : \Omega^q(Y) \to \Omega^q(Y)$ is defined by $\Delta_Y^q = \left(d_q^Y\right)^* d_q^Y + d_{q-1}^Y \left(d_{q-1}^Y\right)^*$. It follows from the Hodge decomposition theorem that

$$\Omega^q(Y) = \mathrm{Im}\, d_{q-1}^Y \oplus \mathcal{H}^q(Y) \oplus \mathrm{Im}\left(d_q^Y\right)^*, \tag{2.2}$$

where

$$\mathcal{H}^q(Y) \;=\; \ker \Delta_Y^q \;=\; \{\omega_Y \in \Omega^q(Y) \mid d_q^Y \omega_Y = \left(d_{q-1}^Y\right)^* \omega_Y = 0\}. \tag{2.3}$$

Suppose that a q-form $\phi \in \Omega^q(M)$ satisfies $d_q\phi = d_{q-1}^*\phi = 0$, where $d_q : \Omega^q(M) \to \Omega^{q+1}(M)$ is the exterior derivative on M and $d_{q-1}^* = (-1)^{mq+m+1} \star_M d_{m-q} \star_M$ is the formal adjoint of d_{q-1}. Simple computation shows that ϕ restricted to $\{0\} \times Y$ is expressed by

$$\begin{aligned} \phi|_Y = \left(d_{q-1}^Y \varphi_1 + \varphi_2\right) \;+\; du \wedge \left(\left(d_q^Y\right)^* \psi_1 + \psi_2\right), \\ \varphi_1, \psi_1 \in \Omega^\bullet(Y), \quad \varphi_2, \psi_2 \in \mathcal{H}^\bullet(Y). \end{aligned} \tag{2.4}$$

In other words, φ_2 and $\star_Y\psi_2$ are the harmonic parts of $\iota^*\phi$ and $\iota^*\big(\star_M \phi\big)$, where $\iota : Y \to M$ is the natural inclusion. We define

$$\mathcal{K}^q = \{\varphi_2 \in \mathcal{H}^q(Y) \mid d_q\phi = (d_{q-1})^*\phi = 0\}, \quad \mathcal{K} = \oplus_{q=0}^{m-1} \mathcal{K}^q. \tag{2.5}$$

Replacing ϕ with $\star_M\phi$ yields $\star_Y\psi_2 \in \mathcal{K}^{m-q}$, which leads to the following result.

$$\begin{aligned} \star_Y \mathcal{K}^{m-q} = \{\psi_2 \in \mathcal{H}^q(Y) \mid d_q\phi = (d_{q-1})^*\phi = 0\}, \\ \star_Y\mathcal{K} = \oplus_{q=0}^{m-1} \star_Y \mathcal{K}^q. \end{aligned} \tag{2.6}$$

In the next two lemmas, we are going to show that $\mathcal{K}$ is exactly the half of $\mathcal{H}^\bullet(Y)$ and $\mathcal{K} \oplus \star_Y\mathcal{K} = \mathcal{H}^\bullet(Y)$ (cf. Corollary 8.4 in [7]).

Lemma 2.1.

$$\dim \mathcal{K} \;=\; \frac{1}{2} \dim \mathcal{H}^\bullet(Y).$$

Proof. We note the following long exact sequence.

$$\longrightarrow H^q(M,Y) \xrightarrow{\alpha_q} H^q(M) \xrightarrow{\iota_q^*} H^q(Y) \xrightarrow{\beta_q} H^{q+1}(M,Y) \longrightarrow \tag{2.7}$$

Since $\mathcal{K}$ is a harmonic part of $\iota^*\phi$ for $\phi \in \Omega^\bullet(M)$, it follows that $\dim \mathcal{K} = \sum_{q=0}^{m-1} \dim \mathrm{Im}\, \iota_q^*$. We denote

$$(2.8)\quad H^q_+(M,Y) := H^q(M,Y) \ominus H^q_-(M,Y), \quad H^{q+1}_-(M,Y) := \operatorname{Im}\beta_q,$$

so that $H^q(M,Y) = H^q_+(M,Y) \oplus H^q_-(M,Y)$. It follows that

$$(2.9)\qquad \begin{aligned} \dim H^q(M) &= \dim H^q_+(M,Y) + \dim \operatorname{Im} \iota^*_q, \\ \dim H^q(Y) &= \dim H^{q+1}_-(M,Y) + \dim \operatorname{Im} \iota^*_q, \end{aligned}$$

which leads to

$$(2.10)\qquad \begin{aligned} \sum_{q=0}^{m-1} \dim H^q(Y) &= \sum_{q=0}^{m-1} \Big(\dim H^{q+1}_-(M,Y) + \dim \operatorname{Im} \iota^*_q\Big) \\ &= \sum_{q=0}^{m-1} \Big(\dim H^{q+1}(M,Y) - \dim H^{q+1}_+(M,Y) + \dim \operatorname{Im} \iota^*_q\Big) \\ &= \sum_{q=0}^{m-1} \big(\dim H^{q+1}(M,Y) - \dim H^{q+1}(M) + \operatorname{Im} \iota^*_{q+1} + \dim \operatorname{Im} \iota^*_q\big) \\ &= \sum_{q=1}^{m} \dim H^q(M,Y) - \sum_{q=1}^{m} \dim H^q(M) \\ &\qquad + \sum_{q=1}^{m} \dim \operatorname{Im} \iota^*_q + \sum_{q=0}^{m-1} \dim \operatorname{Im} \iota^*_q. \end{aligned}$$

Since $H^0(M,Y) = 0$ and $H^0(M) = \mathbb{R}$, it follows from the Lefschetz-Poincaré duality that

$$(2.11)\qquad \sum_{q=1}^{m} \dim H^q(M,Y) - \sum_{q=1}^{m} \dim H^q(M) \;=\; 1.$$

Since $\iota^*_0 : H^0(M) \to H^0(Y)$ is an isomorphism, it follows that $\dim \operatorname{Im} \iota^*_0 = 1$, which leads to

$$(2.12)\qquad \begin{aligned} \sum_{q=0}^{m-1} \dim H^q(Y) &= 1 + \sum_{q=1}^{m} \dim \operatorname{Im} \iota^*_q + \sum_{q=0}^{m-1} \dim \operatorname{Im} \iota^*_q \\ &= 2 \sum_{q=0}^{m-1} \dim \operatorname{Im} \iota^*_q. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.2. *$\mathcal{K}$ is orthogonal to $\star_Y \mathcal{K}$, and hence $\mathcal{K} \oplus \star_Y \mathcal{K} \;=\; \mathcal{H}^\bullet(Y)$.*

Proof. For any $\phi \in \Omega^q(M)$ and $\omega \in \Omega^{q+1}(M)$ satisfying $d_q\phi = d_{q-1}^*\phi = 0$ and $d_{q+1}\omega = d_q^*\omega = 0$, it follows that

$$\begin{aligned}
\phi|_Y &= \left(d_{q-1}^Y\varphi_1 + \varphi_2\right) + du \wedge \left(\left(d_q^Y\right)^* \psi_1 + \psi_2\right), \\
&\qquad \varphi_1, \psi_1 \in \Omega^\bullet(Y), \quad \varphi_2, \psi_2 \in \mathcal{H}^\bullet(Y), \\
\omega|_Y &= \left(d_{q-1}^Y\omega_1 + \omega_2\right) + du \wedge \left(\left(d_q^Y\right)^* \eta_1 + \eta_2\right), \\
&\qquad \omega_1, \eta_1 \in \Omega^\bullet(Y), \quad \omega_2, \eta_2 \in \mathcal{H}^\bullet(Y).
\end{aligned} \tag{2.13}$$

Then, $\varphi_2,\ \omega_2 \in \mathcal{K}$ and $\psi_2,\ \eta_2 \in \star_Y\mathcal{K}$. In fact, $\varphi_2 \in \mathcal{K}^q$ and $\eta_2 \in \star_Y\mathcal{K}^{m-1-q}$. It is enough to show that

$$\langle \varphi_2,\ \eta_2\rangle_Y \ := \ \int_Y \varphi_2 \wedge \star_Y\eta_2 \ = \ 0. \tag{2.14}$$

Since $d_{m-q-1}(\star_M\omega) = 0$, it follows by Stokes' theorem that

$$\begin{aligned}
0 &= \int_M d_q\phi \wedge \star_M\omega = \int_M d_{m-1}\big(\phi \wedge \star_M\omega\big) = \int_Y \iota^*\phi \wedge \iota^*\big(\star_M \omega\big) \\
&= \int_Y \left(d_{q-1}^Y\varphi_1 + \varphi_2\right) \wedge \star_Y \left(\left(d_q^Y\right)^* \eta_1 + \eta_2\right) = \int_Y \varphi_2 \wedge \star_Y\eta_2,
\end{aligned} \tag{2.15}$$

which completes the proof of the lemma. $\square$

Near the boundary Y, a q-form $\omega \in \Omega^q(M)$ can be expressed by $\omega = \omega_1 + du \wedge \omega_2$, where $\iota_{\frac{\partial}{\partial u}}\omega_1 = \iota_{\frac{\partial}{\partial u}}\omega_2 = 0$. We denote ω by

$$\omega = \omega_1 + du \wedge \omega_2 = (\omega_1,\ \omega_2)\,. \tag{2.16}$$

We denote $\mathcal{L}_0 = (\mathcal{K},\ \mathcal{K})$ and $\mathcal{L}_1 = (\star_Y\mathcal{K},\ \star_Y\mathcal{K})$ and define the orthogonal projections

$$\begin{aligned}
&\mathcal{P}_{-,\mathcal{L}_0}, \mathcal{P}_{+,\mathcal{L}_1} : \Omega^\bullet(Y) \oplus \Omega^\bullet(Y) \ \to \ \Omega^\bullet(Y) \oplus \Omega^\bullet(Y), \\
&\qquad \operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0} = \left(\operatorname{Im} d^Y \oplus \mathcal{K},\ \operatorname{Im} d^Y \oplus \mathcal{K}\right), \\
&\qquad \operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1} = \left(\operatorname{Im}(d^Y)^* \oplus \star_Y\mathcal{K},\ \operatorname{Im}(d^Y)^* \oplus \star_Y\mathcal{K}\right).
\end{aligned} \tag{2.17}$$

It is straightforward that

$$\Omega^\bullet(Y) \oplus \Omega^\bullet(Y) = \operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0} \oplus \operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1},\ \star_Y \operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0} = \operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1}, \tag{2.18}$$

which shows that $\Omega^\bullet(Y) \oplus \Omega^\bullet(Y)$ is a symplectic vector space with Lagrangian subspaces $\operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0}$ and $\operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1}$. We define an involution $\Gamma : \Omega^q(M) \to \Omega^q(M)$ by

$$\Gamma\omega := i^{[\frac{m+1}{2}]}(-1)^{\frac{q(q+1)}{2}} \star_M \omega. \tag{2.19}$$

Then, Γ satisfies $\Gamma^2 = \mathrm{Id}$. We define the odd signature operator by

$$\mathcal{D}_M : \Omega^{\bullet}(M) \to \Omega^{\bullet}(M), \qquad \mathcal{D}_M = d\Gamma + \Gamma d. \tag{2.20}$$

It is straightforward that

$$\mathcal{D}_M^2 \;=\; \Delta_M^q \;=\; d_q^* d_q + d_{q-1} d_{q-1}^*. \tag{2.21}$$

It is shown in Lemma 2.15 of [5] that $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ are well posed boundary conditions for the odd signature operator $\mathcal{D}_M$. Then, $\mathcal{D}_M$ with the boundary condition $\mathcal{P}_{-,\mathcal{L}_0}$ or $\mathcal{P}_{+,\mathcal{L}_1}$ has a discrete spectrum. In particular, Δ_M^q with the boundary condition $\mathcal{P}_{-,\mathcal{L}_0}$ or $\mathcal{P}_{+,\mathcal{L}_1}$ has a discrete spectrum. We define

$$\begin{aligned} \Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) &= \left\{ \phi \in \Omega^q(M) \mid \mathcal{P}_{-,\mathcal{L}_0}\left((\mathcal{D}_M^{\ell}\phi)|_Y \right) = 0, \ell = 0,1,2,\cdots \right\}, \\ \Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) &= \left\{ \phi \in \Omega^q(M) \mid \mathcal{P}_{+,\mathcal{L}_1}\left((\mathcal{D}_M^{\ell}\phi)|_Y \right) = 0, \ell = 0,1,2,\cdots \right\}. \end{aligned} \tag{2.22}$$

For example, an eigenform ω of Δ_M^q satisfying $\mathcal{P}_{-,\mathcal{L}_0}(\omega) = \mathcal{P}_{-,\mathcal{L}_0}(\mathcal{D}_M\omega) = 0$ ($\mathcal{P}_{+,\mathcal{L}_1}(\omega) = \mathcal{P}_{+,\mathcal{L}_1}(\mathcal{D}_M\omega) = 0$) belongs to $\Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M)$ ($\Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M)$). Simple computation shows that for each q the exterior derivative d maps $\Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M)$ and $\Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M)$ into $\Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M)$ $\Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M)$, respectively, *i.e.*

$$d : \Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \;\to\; \Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M), \quad d : \Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) \;\to\; \Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M).$$

Keeping these facts in mind, we define two de Rham comlexes as follows.

$$\begin{aligned} (\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M), d) : 0 &\longrightarrow \Omega^{0,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \xrightarrow{d} \Omega^{1,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) \\ &\xrightarrow{d} \Omega^{2,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \xrightarrow{d} \cdots \longrightarrow 0. \\ (\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_1}(M), d) : 0 &\longrightarrow \Omega^{0,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) \xrightarrow{d} \Omega^{1,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \\ &\xrightarrow{d} \Omega^{2,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) \xrightarrow{d} \cdots \longrightarrow 0. \end{aligned} \tag{2.23}$$

We define two Hodge Laplacians $\Delta^q_{M,\widetilde{\mathcal{P}}_0}$ and $\Delta^q_{M,\widetilde{\mathcal{P}}_1}$ with respect to $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$ by

$$(2.24)\qquad \begin{aligned} \Delta^q_{M,\widetilde{\mathcal{P}}_0} &:= d_q^* d_q + d_{q-1} d_{q-1}^*, \\ &\qquad \mathrm{Dom}\left(\Delta^q_{\widetilde{\mathcal{P}}_0}\right) = \Omega^{q,\infty}_{\widetilde{\mathcal{P}}_0}(M) := \begin{cases} \Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) & \text{for } q \text{ even} \\ \Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) & \text{for } q \text{ odd.} \end{cases} \\ \Delta^q_{M,\widetilde{\mathcal{P}}_1} &:= d_q^* d_q + d_{q-1} d_{q-1}^*, \\ &\qquad \mathrm{Dom}\left(\Delta^q_{\widetilde{\mathcal{P}}_1}\right) = \Omega^{q,\infty}_{\widetilde{\mathcal{P}}_1}(M) := \begin{cases} \Omega^{q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) & \text{for } q \text{ even} \\ \Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) & \text{for } q \text{ odd.} \end{cases} \end{aligned}$$

The following lemma is straightforward (see Lemma 2.4 in [6]).

LEMMA 2.3. *The cohomologies of the complex* $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d)$ *are given as follows.*

$$H^q((\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M),\ d)) = \ker \Delta^q_{\widetilde{\mathcal{P}}_0} = \begin{cases} H^q(M,Y) & \textit{if } q \textit{ is even} \\ H^q(M) & \textit{if } q \textit{ is odd,} \end{cases}$$

$$H^q((\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_1}(M),\ d)) = \ker \Delta^q_{\widetilde{\mathcal{P}}_1} = \begin{cases} H^q(M) & \textit{if } q \textit{ is even} \\ H^q(M,Y) & \textit{if } q \textit{ is odd.} \end{cases}$$

By analyzing the heat traces $\sum_{q=0}^m (-1)^q \operatorname{Tr}\left(f^* e^{-t\Delta^q_{M,\widetilde{\mathcal{P}}_0}}\right)$ and $\sum_{q=0}^m (-1)^q \operatorname{Tr}\left(f^* e^{-t\Delta^q_{M,\widetilde{\mathcal{P}}_1}}\right)$, the following is obtained, which is the main result of [6].

THEOREM 2.4. *Let* (M, Y, g^M) *and* $f : M \to M$ *be as above. On a collar neighborhood* U *of* Y *and for* $0 \le c \in \mathbb{R}$, $c \ne 1$, *we assume that* $f(u,y) = (cu + u^2\kappa(u), B(y))$ *for some smooth function* $\kappa : [0,\epsilon) \to [0,\epsilon)$, *and* $B(y)^* : \Omega^\bullet(Y) \to \Omega^\bullet(Y)$ *maps* $\operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0}$ *and* $\operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1}$ *into* $\operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0}$ *and* $\operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1}$, *respectively. Then,*

$$
\begin{aligned}
(1)\ \sum_{q=\text{even}} &\operatorname{Tr}\left(f^* : H^q(M,Y) \to H^q(M,Y)\right) \\
&\qquad - \sum_{q=\text{odd}} \operatorname{Tr}\left(f^* : H^q(M) \to H^q(M)\right) \\
&= \sum_{x \in \mathcal{F}_0(f)} \operatorname{sign}\det(I - df(x)) \\
&\qquad + \frac{1}{2}\operatorname{sign}(1-c) \sum_{y \in \mathcal{F}_Y(f)} \operatorname{sign}\det(I - df_Y(y)) + \frac{1}{2}k_0,
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \sum_{q=\text{even}} &\operatorname{Tr}\left(f^* : H^q(M) \to H^q(M)\right) \\
&\qquad - \sum_{q=\text{odd}} \operatorname{Tr}\left(f^* : H^q(M,Y) \to H^q(M,Y)\right) \\
&= \sum_{x \in \mathcal{F}_0(f)} \operatorname{sign}\det(I - df(x)) \\
&\qquad + \frac{1}{2}\operatorname{sign}(1-c) \sum_{y \in \mathcal{F}_Y(f)} \operatorname{sign}\det(I - df_Y(y)) - \frac{1}{2}k_0,
\end{aligned}
$$

where $k_0 = \operatorname{Tr}\Big(B^* : \star_Y \mathcal{K} \to \star_Y \mathcal{K}\Big) \ - \ \operatorname{Tr}\Big(B^* : \mathcal{K} \to \mathcal{K}\Big)$.

Remark: (1) Theorem 2.4 was proved in [6] when $\kappa(u) = 0$, $c > 1$ and $B(y) : Y \to Y$ is a local isometry. However, the theorem can be easily extended to this case.
(2) So far, we do not know how to extend Theorem 2.4 to a wider class of smooth functions.

3. Examples of Lefschetz fixed point formula on the complex $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d)$

In this section, we are going to give two examples showing Theorem 2.4. The first one is the following. Let $S^1 = \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ be the round circle and $h : S^1 \to S^1$ be defined by $h(e^{i\theta}) = e^{ki\theta}$ for $2 \leq k \in \mathbb{N}$. Then, h has $k-1$ fixed points at $\Big\{e^{\frac{2\pi\ell}{k-1}i} \mid \ell = 0, 1, \cdots, k-2\Big\}$. We note that for $y_\ell = e^{\frac{2\pi\ell}{k-1}i}$, $dh(x_\ell) : T_{y_\ell}S^1 \to T_{y_\ell}S^1$ is given by $dh(y_\ell)(v) = kv$.

For $1 \leq n \in \mathbb{N}$, we denote by $T^n = S^1 \times \cdots \times S^1$ the n-th product of S^1 and give the usual flat product metric on T^n. For $(k_1, \cdots, k_n) \in \mathbb{N}^n$ with $k_j \geq 2$, we define

$$f : T^n \to T^n, \quad f(e^{i\theta_1}, \cdots, e^{i\theta_n}) = (e^{ik_1\theta_1}, \cdots, e^{ik_n\theta_n}). \tag{3.1}$$

Then, f has $(k_1 - 1) \cdots (k_n - 1)$ fixed points. At each fixed point $y \in T^n$, it follows that

$$df(y) : T_yT^n \;\to\; T_yT^n, \quad df(y) = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & k_n \end{pmatrix}, \tag{3.2}$$

which shows that

$$\operatorname{sign}(\operatorname{Id} - df(y)) = (-1)^n. \tag{3.3}$$

We denote $M = [0,1] \times T^n$ with the usual flat product metric. Then, the boundary is $Y = \{0\} \times T^n \cup \{1\} \times T^n$. We choose a smooth function $\rho : [0,1] \to [0,1]$ such that ρ has exactly 3 fixed points at 0, u_0, 1 with $0 < u_0 < 1$ and

$$\begin{aligned} &\rho(0) = 0, \quad \rho(u_0) = u_0, \quad \rho(1) = 1, \\ &0 \leq \rho'(0) < 1, \quad \rho'(u_0) > 1, \quad 0 \leq \rho'(1) < 1. \end{aligned} \tag{3.4}$$

We define

$$F : [0,1] \times T^n \to [0,1] \times T^n, \qquad F(u, y) = (\rho(u), f(y)). \tag{3.5}$$

Then, F has $3(k_1 - 1) \cdots (k_n - 1)$ fixed points. There are $(k_1 - 1) \cdots (k_n - 1)$ interior fixed points and $2(k_1 - 1) \cdots (k_n - 1)$ attracting boundary fixed points. At the interior fixed points, it follows that

$$\begin{aligned} \sum_{(u_0, y) \in \mathcal{F}_0(F)} \operatorname{sign}(\operatorname{Id} - dF(u_0, y)) &= - \sum_{y \in \mathcal{F}_0(f)} \operatorname{sign}(\operatorname{Id} - df(y)) \\ &= (-1)^{n+1}(k_1 - 1) \cdots (k_n - 1). \end{aligned} \tag{3.6}$$

At the boundary fixed points, it follows that

$$\begin{aligned} \sum_{(0,y),(1,y) \in \mathcal{F}_Y^+(F)} \operatorname{sign}(\operatorname{Id} - dF(u_1, y)) &= 2 \sum_{y \in \mathcal{F}_0(f)} \operatorname{sign}(\operatorname{Id} - df(y)) \\ &= (-1)^n 2(k_1 - 1) \cdots (k_n - 1), \end{aligned} \tag{3.7}$$

where $u_1 = 0$ or 1. Hence, it follows that

(3.8)

$$\sum_{(u,y)\in\mathcal{F}_0(F)\cup\mathcal{F}_Y^+(F)} \operatorname{sign}(\mathrm{Id} - dF(u,y)) = (-1)^n(k_1-1)\cdots(k_n-1),$$

$$\begin{aligned}\sum_{(u,y)\in\mathcal{F}_0(F)\cup\mathcal{F}_Y^-(F)} \operatorname{sign}(\mathrm{Id} - dF(u,y)) &= -\sum_{y\in\mathcal{F}_0(f)} \operatorname{sign}(\mathrm{Id} - df(y)) \\ &= (-1)^{n+1}(k_1-1)\cdots(k_n-1).\end{aligned}$$

We are now going to compute $H^q(M)$ by using the de Rham complex. We consider the de Rham complex

$$(3.9) \qquad \to \ \Omega^{q-1}(M) \ \overset{d_{q-1}}{\to} \ \Omega^q(M) \ \overset{d_q}{\to} \ \Omega^{q+1}(M) \ \to$$

A q-form $\omega \in \Omega^q(M)$ can be expressed by $\omega = \omega_1 + du \wedge \omega_2$, where $\iota_{\frac{\partial}{\partial u}}\omega_1 = \iota_{\frac{\partial}{\partial u}}\omega_2 = 0$ with $\frac{\partial}{\partial u}$ the unit vector field normal to Y. Let $\iota : Y \to M$ be the natural inclusion. If $\omega_2|_Y = \iota^*\omega_2 = 0$, then ω is said to satisfy the absolute boundary condition. If $\omega_1|_Y = \iota^*\omega = 0$, then ω is said to satisfy the relative boundary condition. Let $\Omega^q_{\mathrm{nor}}(M)$ and $\Omega^q_{\tan}(M)$ be the space of all smooth q-forms satisfying the absolute and relative boundary conditions, respectively. Then, it is well known (for example, Theorem 2.7.3 in [4]) that

$$(3.10) \qquad \begin{aligned} H^q(M) \ &\cong \ \mathcal{H}^q(M) := \{\omega \in \Omega^q_{\mathrm{nor}}(M) \mid d\omega = d^*\omega = 0\}, \\ H^q(M,Y) \ &\cong \ \mathcal{H}^q(M,Y) := \{\omega \in \Omega^q_{\tan}(M) \mid d\omega = d^*\omega = 0\}. \end{aligned}$$

We denote

$$(3.11) \qquad d\vartheta_j \ = \ (0,0,\cdots,d\theta,0,\cdots,0) \in \Omega^1(M).$$

Then, $\{d\vartheta_{j_1}\wedge\cdots\wedge d\vartheta_{j_q} \mid 1 \le j_1 < \cdots < j_q \le n\}$ is an orthogonal basis of $\mathcal{H}^q(M)$ and $\{du \wedge d\vartheta_{j_1}\wedge\cdots\wedge d\vartheta_{j_{q-1}} \mid 1 \le j_1 < \cdots < j_{q-1} \le n\}$ is an orthogonal basis of $\mathcal{H}^q(M,Y)$, which leads to the following facts.

Lemma 3.1.

$$H^q(M) \cong \mathbb{R}^{\binom{n}{q}} \cong \mathbb{R}^{\frac{n!}{q!(n-q)!}}, \qquad H^q(M,Y) \cong \mathbb{R}^{\binom{n}{q-1}} \cong \mathbb{R}^{\frac{n!}{(q-1)!(n-q+1)!}},$$

$$F^* : H^q(M) \to H^q(M),$$

$$F^*(d\vartheta_{j_1}\wedge\cdots\wedge d\vartheta_{j_q}) = k_{j_1}\cdots k_{j_q} d\vartheta_{j_1}\wedge\cdots\wedge d\vartheta_{j_q},$$

$$F^* : H^q(M,Y) \to H^q(M,Y),$$

$$F^*(du\wedge d\vartheta_{j_1}\wedge\cdots\wedge d\vartheta_{j_{q-1}}) = k_{j_1}\cdots k_{j_{q-1}} du\wedge d\vartheta_{j_1}\wedge\cdots\wedge d\vartheta_{j_{q-1}}.$$

Lemma 3.1 leads to the following result.

(3.12)

$$\sum_{q=0}^{n+1}(-1)^q \operatorname{Tr}\left(F^* : \mathcal{H}^q(M) \to \mathcal{H}^q(M)\right) = \sum_{q=0}^{n}(-1)^q \sum_{1\leq j_1<\cdots<j_q\leq n} k_{j_1}\cdots k_{j_q}$$
$$= (1-k_1)(1-k_2)\cdots(1-k_n) \ = \ (-1)^n(k_1-1)(k_2-1)\cdots(k_n-1),$$
$$\sum_{q=0}^{n+1}(-1)^q \operatorname{Tr}\left(F^* : \mathcal{H}^q(M,Y) \to \mathcal{H}^q(M,Y)\right)$$
$$= \sum_{q=0}^{n+1}(-1)^q \sum_{1\leq j_1<\cdots<j_{q-1}\leq n} k_{j_1}\cdots k_{j_{q-1}}$$
$$= -\sum_{q=0}^{n}(-1)^q \sum_{1\leq j_1<\cdots<j_q\leq n} k_{j_1}\cdots k_{j_q} = -(1-k_1)(1-k_2)\cdots(1-k_n)$$
$$= (-1)^{n+1}(k_1-1)(k_2-1)\cdots(k_n-1).$$

Eq.(3.12) together with (3.8) shows (1.5).

On the other hand, since $c = \rho'(0)$ or $\rho'(1)$, it follows that $0 < c < 1$. Hence,

(3.13)

$$\sum_{(u_0,y)\in\mathcal{F}_0(F)} \operatorname{sign}\det(I - dF(u_0,y)) + \frac{1}{2}\sum_{y\in\mathcal{F}_0(f)} \operatorname{sign}\det(I-df(y))$$
$$= (-1)^{n+1}(k_1-1)\cdots(k_n-1)$$
$$+ \frac{1}{2}\cdot 2\cdot(-1)^n(k_1-1)\cdots(k_n-1) = 0.$$

We note that

(3.14)

$$\sum_{q=\mathrm{even}} \operatorname{Tr}\left(F^* : H^q(M,Y) \to H^q(M,Y)\right) = \sum_{q=\mathrm{even}}\ \sum_{1\leq j_1<\cdots<j_{q-1}\leq n} k_{j_1}\cdots k_{j_{q-1}}$$
$$= \sum_{q=\mathrm{odd}}\ \sum_{1\leq j_1<\cdots<j_q\leq n} k_{j_1}\cdots k_{j_q} = \sum_{q=\mathrm{odd}} \operatorname{Tr}\left(F^* : H^q(M) \to H^q(M)\right),$$
$$\sum_{q=\mathrm{odd}} \operatorname{Tr}\left(F^* : H^q(M,Y) \to H^q(M,Y)\right) = \sum_{q=\mathrm{odd}}\ \sum_{1\leq j_1<\cdots<j_{q-1}\leq n} k_{j_1}\cdots k_{j_{q-1}}$$
$$= \sum_{q=\mathrm{even}}\ \sum_{1\leq j_1<\cdots<j_q\leq n} k_{j_1}\cdots k_{j_q} = \sum_{q=\mathrm{even}} \operatorname{Tr}\left(F^* : H^q(M) \to H^q(M)\right),$$

which shows that

$$(3.15)\quad \begin{aligned}&\sum_{q=\text{even}} \mathrm{Tr}\,(F^* : H^q(M,Y) \to H^q(M,Y)) \\ &\qquad - \sum_{q=\text{odd}} \mathrm{Tr}\,(F^* : H^q(M) \to H^q(M)) = 0, \\ &\sum_{q=\text{even}} \mathrm{Tr}\,(F^* : H^q(M) \to H^q(M)) \\ &\qquad - \sum_{q=\text{odd}} \mathrm{Tr}\,(F^* : H^q(M,Y) \to H^q(M,Y)) = 0.\end{aligned}$$

We finally consider $\mathcal{K}^q$ and $\star_Y \mathcal{K}^q$. We note that $M = [0,1] \times T^n$ and $Y = \{0\} \times T^n \cup \{1\} \times T^n$. We denote $\iota_0 : \{0\} \times T^n \to M$ and $\iota_1 : \{1\} \times T^n \to M$. Then, $(\iota_0, \iota_1) : Y \to M$ is the natural inclusion. The harmonic space $\mathcal{H}^\bullet(M)$ on M is given by

$$(3.16)\quad \begin{aligned}\mathcal{H}^\bullet(M) &= \oplus_{q=0}^n \{d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q} \mid 1 \le j_1 < \cdots < j_q \le n\} \\ &\subset \Omega^\bullet(M).\end{aligned}$$

Then, $\mathcal{K}^q$ is given by

$$(3.17)\quad \begin{aligned}\mathcal{K}^q = &\{(d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q}, d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q}) \mid 1 \le j_1 < \cdots < j_q \le n\} \\ &\subset \Omega^\bullet(T^n) \oplus \Omega^\bullet(T^n).\end{aligned}$$

We note that an orientation of $\{0\} \times T^n$ is opposite to an orientation of $\{1\} \times T^n$. We choose and fix an orientation $d\,\mathrm{vol}(T^n)$ of $\{0\} \times T^n$ induced from an orientation of M. Then, the orientation of $\{1\} \times T^n$ is $-d\,\mathrm{vol}(T^n)$. Hence,

$$(3.18)\quad \star_Y \mathcal{K}^q \;=\; \{(\star_{T^n}\omega,\ -\star_{T^n}\omega) \mid (\omega,\ \omega) \in \mathcal{K}^q\}.$$

Since $B(y) = f(y)$, it follows that

$$(3.19)\quad \begin{aligned}&B^* = f^* : \mathcal{K}^q \;\to\; \mathcal{K}^q, \\ &f^*\left(d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q},\ d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q}\right) \\ &\qquad = k_{j_1} \cdots k_{j_q}\left(d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q},\ d\vartheta_{j_1} \wedge \cdots \wedge d\vartheta_{j_q}\right), \\ &B^* = f^* : \star_Y \mathcal{K}^{n-q} \;\to\; \star_Y \mathcal{K}^{n-q}, \\ &f^*\left(\star_{T^n} d\vartheta_{i_1} \wedge \cdots \wedge d\vartheta_{i_{n-q}},\ -\star_{T^n} d\vartheta_{i_1} \wedge \cdots \wedge d\vartheta_{i_{n-q}}\right) \\ &\qquad = k_{j_1} \cdots k_{j_q}\left(\star_{T^n} d\vartheta_{i_1} \wedge \cdots \wedge d\vartheta_{i_{n-q}},\ -\star_{T^n} d\vartheta_{i_1} \wedge \cdots \wedge d\vartheta_{i_{n-q}}\right),\end{aligned}$$

where $\{i_1, \cdots, i_{n-q}\} = \{1, 2, \cdots, n\} - \{j_1, \cdots, j_q\}$. This shows that

$$(3.20) \qquad k_0 = \operatorname{Tr}\Big(B^* : \star_Y \mathcal{K} \to \star_Y \mathcal{K}\Big) \ - \ \operatorname{Tr}\Big(B^* : \mathcal{K} \to \mathcal{K}\Big) \ = \ 0,$$

which together with (3.13) and (3.15) shows Theorem 2.4.

The second example is the following. We denote $\mathbb{D}^2 = \{z = x + iy \mid x^2 + y^2 \leq 1\} \subset \mathbb{C}$ and define $f : \mathbb{D}^2 \to \mathbb{D}^2$ by $f(z) = z^n$ for $n \in \mathbb{N}$. Then, $f(z)$ has n fixed points, which are $z = 0$ and $z = e^{\frac{2\pi\ell}{n-1}i}$, $\ell = 0, 1, 2, \cdots, n-2$. Simple computation shows that

$$(3.21) \qquad df(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad df(e^{\frac{2\pi\ell}{n-1}i}) = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix},$$

which shows that

$$(3.22) \quad \operatorname{sign}\det(\operatorname{Id} - df(0)) \ = \ \operatorname{sign}\det(\operatorname{Id} - df(e^{\frac{2\pi\ell}{n-1}i})) \ = \ 1.$$

We choose a Riemann metric on $\mathbb{D}^2$, which is a product one on a collar neighborhood $U = (\epsilon_0, 1] \times S^1$ of the boundary. For $\epsilon_1 = \epsilon_0^{\frac{1}{n}}$, it follows that

$$(3.23) \qquad f|_U : (\epsilon_1, 1] \times S^1 \to (\epsilon_0, 1] \times S^1, \qquad f(r, e^{i\theta}) = (r^n, e^{in\theta}),$$

which shows that all the boundary fixed points are repelling. We define $\phi_0 : (\epsilon_0, 1] \to [0, 1 - \epsilon_0)$ by $\phi_0(r) = 1 - r$ and $\phi_1 : (\epsilon_1, 1] \to [0, 1 - \epsilon_1)$ by $\phi_1(r) = 1 - r$. Then, $f|_U$ can be rewritten by

$$\widetilde{f|_U} : [0, 1 - \epsilon_1) \times S^1 \to [0, 1 - \epsilon_1) \times S^1,$$
$$\widetilde{f|_U}(r, e^{i\theta}) = (1 - (1 - r)^n, e^{in\theta}) = (nr + r^2\kappa(u), e^{in\theta}),$$

where $\kappa(u) = \frac{1}{r^2}\Big\{1 - (1 - r)^n - nr\Big\}$. We denote $f_Y := f|_{S^1}$ and note that

$$(3.24) \qquad f_Y : S^1 \to S^1, \qquad f_Y(e^{i\theta}) = e^{in\theta},$$

which shows that for $y_\ell = e^{\frac{2\pi\ell}{n-1}i}$,

$$(3.25) \qquad df_Y(y_\ell) : T_{y_\ell}S^1 \to T_{y_\ell}S^1, \qquad df_Y(y_\ell)(v) = nv,$$

and $\operatorname{sign}\det(\operatorname{Id} - df_Y(y_\ell)) = -1$. Hence, we obtain the following.

$$\sum_{x\in\mathcal{F}_0(f)} \operatorname{sign}\det(\operatorname{Id}-df(x)) = 1, \tag{3.26}$$
$$\sum_{x\in\mathcal{F}_0(f)\cup\mathcal{F}_Y^-(f)} \operatorname{sign}\det(\operatorname{Id}-df(x)) = 1+(n-1) = n,$$
$$\sum_{y\in\mathcal{F}_0(f_Y)} \operatorname{sign}\det(\operatorname{Id}-df_Y(y)) = -(n-1).$$

We note that

$$\mathcal{K} = \{\text{constant functions}\}, \quad \star_Y\mathcal{K} = \{r d\theta \mid r\in\mathbb{R}\}, \tag{3.27}$$

which shows that

$$B^* = f_Y^* : \star_Y\mathcal{K} \to \star_Y\mathcal{K}, f_Y^*(d\theta) = n d\theta, B^* = f_Y^* : \mathcal{K} \to \mathcal{K}, f_Y^*(1) = 1. \tag{3.28}$$

Hence, k_0 defined in Theorem 2.4 is $n-1$. We note that

$$\begin{aligned} &H^0(\mathbb{D}^2) = \mathbb{R}, H^1(\mathbb{D}^2) = H^2(\mathbb{D}^2) = 0, \\ &H^0(\mathbb{D}^2, S^1) = H^1(\mathbb{D}^2, S^1) = 0, H^2(\mathbb{D}^2, S^1) = \mathbb{R}, \\ &f^* : H^0(\mathbb{D}^2) \to H^0(\mathbb{D}^2), f^*(1) = 1, \\ &f^* : H^2(\mathbb{D}^2, S^1) \to H^0(\mathbb{D}^2, S^1), f^*(1) = n. \end{aligned} \tag{3.29}$$

Finally, we obtain the following result.

$$\begin{aligned} &\sum_{q=0}^{2}(-1)^q \operatorname{Tr}\Big(f^* : H^q(\mathbb{D}^2) \to H^q(\mathbb{D}^2)\Big) \\ &\qquad = \sum_{x\in\mathcal{F}_0(f)} \operatorname{sign}\det(\operatorname{Id}-df(x)) = 1, \\ &\sum_{q=0}^{2}(-1)^q \operatorname{Tr}\Big(f^* : H^q(\mathbb{D}^2, S^1) \to H^q(\mathbb{D}^2, S^1)\Big) \\ &\qquad = \sum_{x\in\mathcal{F}_0\cup\mathcal{F}_Y^-(f)} \operatorname{sign}\det(\operatorname{Id}-df(x)) = n, \end{aligned}$$

which shows (1.5). We also note that

$$\begin{aligned}
&\sum_{q=\mathrm{even}}^{2} (-1)^q \operatorname{Tr}\Big(f^* : H^q(\mathbb{D}^2, S^1) \to H^q(\mathbb{D}^2, S^1)\Big) \\
&\qquad - \sum_{q=\mathrm{odd}}^{2} (-1)^q \operatorname{Tr}\Big(f^* : H^q(\mathbb{D}^2) \to H^q(\mathbb{D}^2)\Big) \\
&\quad = \sum_{x\in\mathcal{F}_0(f)} \operatorname{sign}\det(\mathrm{Id} - df(x)) - \frac{1}{2} \sum_{x\in\mathcal{F}_0(f_Y)} \operatorname{sign}\det(\mathrm{Id} - df(x)) + \frac{1}{2}k_0 \\
&\quad = n, \\
&\sum_{q=\mathrm{even}}^{2} (-1)^q \operatorname{Tr}\Big(f^* : H^q(\mathbb{D}^2) \to H^q(\mathbb{D}^2)\Big) \\
&\qquad - \sum_{q=\mathrm{odd}}^{2} (-1)^q \operatorname{Tr}\Big(f^* : H^q(\mathbb{D}^2, S^1) \to H^q(\mathbb{D}^2, S^1)\Big) \\
&\quad = \sum_{x\in\mathcal{F}_0(f)} \operatorname{sign}\det(\mathrm{Id} - df(x)) - \frac{1}{2} \sum_{x\in\mathcal{F}_0(f_Y)} \operatorname{sign}\det(\mathrm{Id} - df(x)) - \frac{1}{2}k_0 \\
&\quad = 1,
\end{aligned}$$

which shows Theorem 2.4.

References

[1] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. I*, Ann. Math., **86** (1967), 374-407.

[2] F. Bei, *The L^2-Atiyah-Bott-Lefschetz theorem on manifolds with conical singularities: a heat kernel approach*, Ann. Glob. Anal. Geom., **44** (2013), 565-605.

[3] V. A. Brenner and M. A. Shubin, *Atiyah-Bott-Lefschetz formula for elliptic complexes on manifolds with boundary*, J. Soviet Math., **64** (1993), 1069-1111.

[4] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, 2nd Edition, CRC Press, Inc., 1994.

[5] R.-T. Huang and Y. Lee, *The refined analytic torsion and a well-posed boundary condition for the odd signature operator*, J. Geom. Phys., **126** (2018), 68-92.

[6] R.-T. Huang and Y. Lee, *Lefschetz fixed point formula on a compact Riemannian manifold with boundary for some boundary conditions*, Geom. Dedicata., **181** (2016), 43-59.

[7] P. Kirk and M. Lesch, *The η-invariant, Maslov index and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math., **16** (2004), no.4, 553-629.

[8] J. Roe, *Elliptic operators, topology and asymptotic methods(2nd Ed.)*, Research Notes in Mathematics series 395, Chapman and Hall/CRC, 1998.

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